ASSIGNMENT - 1

UNIT - 1

Que. 1 Expand $e^{a\sin^{-1}x}$ by Maclaurin's theoram?

Solution ⇒ $f(x) = e^{a\sin^{-1}x}$

put $x=0$

$f(0) = e^{a\sin^{-1}0} = 1 \Rightarrow \boxed{f(0) = 1}$

$f(x) = e^{a\sin^{-1}x}$

diff. w.r.t x

$$f'(x) = a\frac{e^{a\sin^{-1}x}}{\sqrt{1+x^2}}$$

put $x=0$

$$f'(0) = a\frac{e^{a\sin^{-1}0}}{\sqrt{1+0^2}}$$

$\boxed{f'(0) = a}$

$$f'(x) = a.\frac{e^{a\sin^{-1}x}}{\sqrt{1+x^2}}$$

diff. w.r.t x

$$f''(x) = a\left[\frac{\sqrt{1+x^2}\, f'(x) - \frac{f(x)}{\cancel{2}} \times \frac{1}{\sqrt{1+x^2}} . \cancel{2}x}{(1+x^2)}\right]$$

put $x=0$

$$f''(0) = a\left[\frac{\sqrt{1+0^2}\, f'(0) - f(0)\frac{1}{\sqrt{1+0^2}} \times 0}{(1+0^2)}\right]$$

$f''(0) = a \times a = a^2 \Rightarrow \boxed{f''(0) = a^2}$

Date :
P. No. :

by using maclaurin's theoram,

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2} f''(0) + \cdots \quad \text{—— (1)}$$

Put the value of $f(0)$, $f'(0)$ and $f''(0)$ in eq. (1)

$$e^{a\sin^{-1}x} = 1 + x \times a + \frac{x^2}{2} \times a^2 + \cdots$$

$$\boxed{e^{a\sin^{-1}x} = 1 + ax + \frac{x^2}{2} a^2 + \cdots}$$ **Ans.**

Que. 2 If $u = x^3 y^2 (1-x-y)$ then discuss the maximum & minimum value of the function.

Sol.^{n} ⇒

$$U = x^3 y^2 (1-x-y) \quad \text{—— (1)}$$

$$\frac{\partial U}{\partial x} = y^2 \left[3x^2 (1-x-y) + x^3 (-1) \right]$$

$$\frac{\partial U}{\partial x} = y^2 \left[3x^2 - 3x^3 - 3x^2 y - x^3 \right]$$

$$\frac{\partial U}{\partial x} = x^2 y^2 \left[3 - 4x - 3y \right]$$

$$\left[\frac{\partial U}{\partial x} = 0 \right] \Rightarrow 0 = x^2 (y^2) \left[3 - 4x - 3y \right] \quad \text{—— (2)}$$

$$\frac{\partial U}{\partial y} = x^3 \left(2y (1-x-y) + y^2 (-1) \right)$$

$$\frac{\partial U}{\partial y} = x^3 \left(2y - 2xy - 2y^2 - y^2 \right)$$

Date :
P. No. :

$$\frac{\partial u}{\partial y} = x^3(2y - 2xy - 3y^2) = yx^3(2-2x-3y)$$

$$\frac{\partial u}{\partial y} = 0 \Rightarrow 0 = yx^3(2-2x-3y) \quad \text{— (3)}$$

From eq. – (2)

$$\boxed{x = \frac{3-3y}{4}} \quad \text{— (4)}$$

From eq. (3) & (4)

$$2 - \cancel{2}\left(\frac{3-3y}{\cancel{4}\,2}\right) - 3y = 0$$

$$4 - 3 + 3y - 6y = 0$$

$$1 - 3y = 0$$

$$\boxed{y = \frac{1}{3}}$$

[∴ from eq. (4)]

$$x = \frac{3-1}{4} = \frac{1}{2} \Rightarrow \boxed{x = \frac{1}{2}}$$

$$r = \left(\frac{\partial^2 u}{\partial x^2}\right)_{\left(\frac{1}{2},\frac{1}{3}\right)} = y^2\left(2x(3-4x-3y) + x^2(-4)\right)$$

$$r = y^2(6x - 8x^2 - 6xy - 4x^2)$$

$$r = y^2(6x - 6xy - 12x^2)$$

$$r = \frac{1}{9}(3-1-3) = -\frac{1}{9} \Rightarrow \boxed{r = -\frac{1}{9}}$$

$$t = \left(\frac{\partial^2 u}{\partial y^2}\right)_{\left(\frac{1}{2},\frac{1}{3}\right)} = x^3(2-2x-3y) + y(-3) = x^3(2-2x-3y)$$

$$t = \frac{1}{8}(\cancel{2}-1-\cancel{2}) = -\frac{1}{8} \Rightarrow \boxed{t = -\frac{1}{8}}$$

$$s = \left(\frac{\partial^2 u}{\partial x \partial y}\right)_{\left(\frac{1}{2},\frac{1}{3}\right)} = y\left[3x^2(2-2x-3y) + x^3(-2)\right]$$

$$s = y(6x^2 - 8x^3 - 9x^2 y)$$

$$s = \frac{1}{3}\left(\frac{3}{2} - 1 - \frac{3}{4}\right) = -\frac{1}{12} \Rightarrow \boxed{s = -\frac{1}{12}}$$

$$rt - s^2$$

$$-\frac{1}{9} \times \left(-\frac{1}{8}\right) - \left(-\frac{1}{12}\right)^2 = \frac{1}{72} - \frac{1}{144} = \frac{1}{144}$$

[$\because$ $rt - s^2 > 0$ & $r < 0$]
[Maxima at $\left(\frac{1}{2}, \frac{1}{3}\right)$]

from eq. (1) Maxima value of $u = x^3 y^2 (1-x-y)$

$$u = \left(\frac{1}{2}\right)^3 \left(\frac{1}{3}\right)^2 \left(1 - \frac{1}{2} - \frac{1}{3}\right)$$

$$u = \frac{1}{8} \times \frac{1}{9}\left(\frac{6-(3-2)}{6}\right) = \frac{1}{72}\left(\frac{1}{6}\right)$$

$$u = \frac{1}{72}\left(\frac{1}{6}\right) \Rightarrow u = \frac{1}{72}\left(\frac{1}{6}\right) = \boxed{u = \frac{1}{432}}$$

Que. 3 Expand $\log(1+\sin x)$ by Maclaurin's theoram.

Sol.^{n} Let $f(x) \Rightarrow \boxed{y = \log(1+\sin x)}$ — (1)

put $x=0$

$y(0) = 0$ — (2)

diff. w.r.t. x

$$y_1' = \frac{1}{1+\sin x}(\cos x) = \frac{\sin\left(\frac{\pi}{2}-x\right)}{1+\cos\left(\frac{\pi}{2}-x\right)}$$

[∴ Using $\sin 2A = 2\cos A \sin A$
$1+\cos 2A = 2\cos^2 A$]

$$y_1 = \frac{\cancel{2}\sin\left(\frac{1}{2}\left(\frac{\pi}{2}-x\right)\right)\cdot\cancel{\cos\left(\frac{1}{2}\left(\frac{\pi}{2}-x\right)\right)}}{\cancel{2}\cos^{\cancel{2}}\left(\frac{1}{2}\left(\frac{\pi}{2}-x\right)\right)}$$

$$y_1 = \tan\left(\frac{1}{2}\left(\frac{\pi}{2}-x\right)\right) \text{ — (3)}$$

diff. (2) w.r.t. x

$$y_2 = \left(-\frac{1}{2}\right)\sec^2\left(\frac{1}{2}\left(\frac{\pi}{2}-x\right)\right) = \frac{-1}{2}\left[1+\tan^2\left(\frac{1}{2}\left(\frac{\pi}{2}-x\right)\right)\right] \text{ — (4)}$$

from eq. (3) & (4)

$$y_2 = -1(1+y_1^2) \text{ — (5)}$$

diff. eq. 5 w.r.t x

$$y_3 = \frac{-1}{2}(2y_1 \cdot y_2) \quad — 6$$

Diff. 6 w.r.t x

$$y_4 = \frac{-1}{2}\left(2(y_1 y_3 + y_2^2)\right) = -(y_1 y_3 + y_2^2) \quad 7$$

put $x=0$ in eq 3

$y_1(0) = 1$

put $x=0$ in eq. 5

$y_2(0) = -1$

put $x=0$ in eq. 6

$y_3(0) = 1$

put $x=0$ in eq. 7

$y_4(0) = -2$

From Maclaurin's theoram

$$f(x) = f(0) + x f_1(0) + \frac{x^2 f_2(0)}{2!} + \frac{x^3 f_3(0)}{3!} + \dots + \frac{x^n f_n(0)}{n!}$$

$$\log(1+\sin x) = 0 + x(1) + \frac{x^2(-1)}{2!} + \frac{x^3(1)}{3!} + \frac{x^4(-2)}{4!} + \dots$$

$$\log(1+\sin x) = x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{2x^4}{4!} + \dots$$

Que. 4 Expand $\tan\left(x+\frac{\pi}{4}\right)$ as far as x^4 term and evaluate $\tan(46.5)$ significant to four digits.

Sol.$^n \Rightarrow$

Let $f(x) = \tan x$ — (1) ; $\boxed{f\left(\frac{\pi}{4}\right) = \tan\frac{\pi}{4} = 1}$

Diff w.r.t x

$f_1'(x) = \sec^2 x = 1+\tan^2 x = 1+(f(x))^2$ (2) ; $\boxed{f_1'\left(\frac{\pi}{4}\right) = 1+1 = 2}$

Diff w.r.t x

$f_2(x) = 2 f(x)\cdot f_1(x)$ — (3) ; $\boxed{f_2\left(\frac{\pi}{4}\right) = 4}$

Diff w.r.t x

$f_3(x) = 2\left(f_1(x)\right)^2 + 2 f(x)\cdot f_2(x)$; $\boxed{f_3\left(\frac{\pi}{4}\right) = 8+8 = 16}$

Diff. w.r.t x

$f_4(x) = 4\cdot f_1(x)\cdot f_2(x) + 2 f_2(x)\cdot f_1(x) + 2 f(x)\cdot f_3(x)$

$f_4(x) = 6\cdot f_1(x)\cdot f_2(x) + 2 f_3(x)\, f(x)$

$$\boxed{f_4\left(\frac{\pi}{4}\right) = 48+32 = 80}$$

By taylor's theoram,

$f(a+h) = f(a) + h\, f_1(a) + \frac{h^2}{2!} f_2(a) + \frac{h^3}{3!} f_3(a) + \ldots$

Date :
P. No. :

Putting $x=h$ & $a=\frac{\pi}{4}$ in above taylor theoram

$$f\left(x+\frac{\pi}{4}\right) = 1 + x(2) + \frac{x^2}{2!}(4) + \frac{x^3}{3!}(16) + \frac{x^4}{4!}(80) + \ldots$$

$$\boxed{f\left(x+\frac{\pi}{4}\right) = 1 + 2x + 2x^2 + \frac{8x^3}{3} + \frac{10x^4}{3} + \ldots}$$

Putting $\boxed{x = 1.5}$

$$f(1.5+45^\circ) = 1 + 2(1.5) + 2(1.5)^2 + \frac{8(1.5)^3}{3} + \frac{(1.5)^4 \times 10}{3} + \ldots$$

$$f(46.5^\circ) = 1 + \frac{2\pi}{(120)} + \frac{2\pi^2}{(120)^2} + \frac{8\pi^3}{(120)^3 \times 3} + \frac{10\pi^4}{(120)^4 \times 3}$$

$$f(46.5^\circ) = 1.05378$$

$$\boxed{\tan(46.5) = 1.05}$$ Ans.

Que. 5 Use Taylor's theoram & prove that

$$\tan^{-1}(x+h) = \tan^{-1}x + h\sin^2\theta - h^2\sin^2\theta\,\frac{\sin 2\theta}{2} + (h\sin\theta)^3\frac{\sin 3\theta}{3} + \ldots (-1)^{n-1}(h\sin\theta)^n\frac{\sin n\theta}{n}$$

where $\theta = \cot^{-1}x$.

Sol.^{n} ⇒ We have to expand $\tan^{-1}(x+h)$ in power of h.

By Taylor's theoram,

$$f(a+h) = f(x+h) = f(x) + hf_1(x) + \frac{h^2}{2!} f_2(x) + \dots + \frac{h^n}{n!} f^n(x) \quad \text{--- (1)}$$

$\boxed{a = x}$

We have to prove that

$$\tan^{-1}(x+h) = \tan^{-1}x + h\sin\theta^2 - h^2\sin^2\theta \cdot \frac{\sin 2\theta}{2} + \dots + (-1)^{n-1}(h\sin\theta)^n \frac{\sin n\theta}{n} \quad \text{--- (2)}$$

Let $f(x) = \tan^{-1}x$

from eq. (1) & eq. (2)

$$f^n(x) = (-1)^{n-1}(\sin\theta)^n \cdot \sin n\theta \, (n-1)!$$

putting $n = 1, 2, 3, \dots$

$$f_1(x) = \sin\theta \cdot \sin\theta = \sin^2\theta \quad \text{--- (3)}$$

$$f_2(x) = -\sin^2\theta \, \sin 2\theta \quad \text{--- (4)}$$

$$f_3(x) = (\sin\theta)^3 \cdot \sin 3\theta \, 2! \quad \text{--- (5)}$$

from eq (1), (2), (3), (4) and (5)

$$f(x+h) = \tan^{-1}(x+h) = \tan^{-1}x + h\sin^2\theta - \frac{(h\sin\theta)^2}{2!}(\sin 2\theta)$$

$$+ (h\sin\theta)^3 \cdot \frac{\sin 3\theta}{3!}(2!) + \dots + (-1)^{n-1}(h\sin\theta)^n \frac{\sin n\theta}{n}$$

$$\boxed{\tan^{-1}(x+h) = \tan^{-1}x + h\sin^2\theta + (h(\sin\theta))^2 \frac{\sin 2\theta}{2} + \dots + (-1)^{n-1}(h\sin\theta)^n \frac{\sin n\theta}{n}}$$

Hence Proved!

Date :
P. No. :

<u>Que.6</u> Find the radius of curvature at the point (P), the given cycloid is –

$$x = a(t + \sin t)$$
$$y = a(1 - \cos t)$$

<u>Sol.ⁿ</u>

$x = a(t + \sin t)$ | $y = a(1 - \cos t)$

diff. w.r.t (t) | diff. w.r.t (t)

$$\boxed{\frac{dx}{dt} = a(1+\cos t)} \text{ — (1)} \qquad \boxed{\frac{dy}{dt} = a\sin t} \text{ — (2)}$$

eq. (2) divided by eq. (1)

$$\frac{\frac{dy}{\cancel{dt}}}{\frac{dx}{\cancel{dt}}} = \frac{a\sin t}{a(1+\cos t)} \Rightarrow \frac{dy}{dx} = \frac{\cancel{a}\sin t}{\cancel{a}(1+\cos t)}$$

$$\frac{dy}{dx} = \frac{\cancel{2}\sin\frac{t}{2}\cdot\cancel{\cos\frac{t}{2}}}{\cancel{2}\cos^{\cancel{2}}\frac{t}{2}}$$

$$\left[\therefore \begin{array}{l} \sin 2\theta = 2\sin\theta\cos\theta \\ \cos 2\theta = 2\cos^2\theta - 1 \end{array}\right]$$

$$\boxed{\frac{dy}{dx} = \tan\left(\frac{t}{2}\right)} \text{ — (3)}$$

Diff. eq. (3) w.r.t x

$$\frac{d^2y}{dx^2} = \frac{1}{2}\sec^2\left(\frac{t}{2}\right)\cdot\frac{dt}{dx}$$

from eq (1)

$$\frac{d^2y}{dx^2} = \frac{1}{2}\sec^2\left(\frac{t}{2}\right)\times\frac{1}{a(1+\cos t)}$$

$$\frac{d^2y}{dx^2} = \frac{1}{2} \times \frac{\sec^2\left(\frac{t}{2}\right)}{a \cdot 2\cos^2\left(\frac{t}{2}\right)}$$

$$\frac{d^2y}{dx^2} = \frac{1}{2} \times \frac{1}{2a} \times \sec^4\left(\frac{t}{2}\right)$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{\sec^4(t/2)}{4a}} \quad (4)$$

- <u>Radius of curvature :-</u>

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

$$\rho = \frac{\left[1 + \tan^2(t/2)\right]^{3/2}}{\frac{\sec^4\left(\frac{t}{2}\right)}{4a}}$$

$$\rho = 4a \times \frac{1}{\sec\left(\frac{t}{2}\right)} = 4a\cos\left(\frac{t}{2}\right)$$

$[\because t = 0]$ $\qquad$ $[\because \cos 0^\circ = 1]$

$$\boxed{\rho = 4a}$$

Que. 7 Find the co-ordinates of centre of curvature at the point (x, y) of the ellispe $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Sol.^{n} ⇒ $$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1} \quad \text{— (1)}$$

Diff. w.r.t x

$$\frac{2x}{a^2} + 2y \cdot \frac{dy}{dx} \cdot \left(\frac{1}{b^2}\right) = 0$$

$$\frac{dy}{dx} = \frac{-\cancel{2}x b^2}{a^2 \cancel{2}y}$$

$$\boxed{\frac{dy}{dx} = \frac{-xb^2}{a^2y}} \quad \text{— (2)}$$

diff. w.r.t x

$$\frac{d^2y}{dx^2} = \frac{-b^2}{a^2}\left[\frac{y \cdot 1 - x\frac{dy}{dx}}{y^2}\right]$$

$$\frac{d^2y}{dx^2} = \frac{-b^2}{a^2y^2}\left[y - x\left[\frac{-xb^2}{a^2y}\right]\right]$$

$$\frac{d^2y}{dx^2} = \frac{-b^2}{a^2y^2}\left[\frac{a^2y^2 + x^2b^2}{a^2y}\right]$$

from eq.^{n} (1) $\quad b^2x^2 + a^2y = a^2 \cdot b^2$

(∴ by both side by $a^2 \cdot b^2$)

Date :
P. No. :

$$\frac{d^2y}{dx^2} = \frac{-b^2}{a^2y^2}\left[\frac{\cancel{a^2}b^2}{\cancel{a^2}y}\right]$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{-b^4}{a^2y^3}} \quad (3)$$

at the point (x, y) the coordinates of centre of curvature –

$$\alpha = \frac{x - \left(\frac{dy}{dx}\right)\left[1 + \left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}}$$

$$\alpha = \frac{x - \left(\frac{-xb^2}{a^2y}\right)\left[1 + \left(\frac{-xb^2}{a^2y}\right)^2\right]}{\frac{-b^4}{a^2y^3}}$$

From eq. (2) & (3)

$$\alpha = \frac{x + \frac{xb^2}{a^2y}\left(1 + \frac{x^2b^4}{a^4y^2}\right)}{\frac{-b^4}{a^2y^3}}$$

$$\alpha = x - \frac{\cancel{a^2}\cancel{y^3}x\cancel{b^2}}{a^2\cancel{y}\cancel{b^4}b^2}\left(\frac{a^4 y^2 + x^2b^4}{a^2\cancel{a^4}\cancel{y^2}}\right)$$

$$\alpha = x - \frac{x(a^4y^2 + x^2b^4)}{a^4b^2} = x - \frac{(b^4x^2 + a^4y^2)x}{a^4b^2} \quad (4)$$

Date :
P. No. :

$$\beta = y + \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}}$$

$$\beta = y + \frac{\left[1 + \left(-\frac{xb^2}{a^2y}\right)^2\right]}{\left(-b^4/a^2y^3\right)}$$

$$\beta = y + \frac{a^2y^3\left[\frac{a^4y^2 + x^2b^4}{a^4y^2}\right]}{-b^4}$$

$$\beta = y - \cancel{a^2}y^{\cancel{3}}\left(\frac{a^4y^2 + x^2b^4}{\cancel{a^4}\,\cancel{y^2}\,b^4}\right)$$

$$\boxed{\beta = y - \frac{(b^4x^2 + a^4y^2)\,y}{a^2b^4}} \quad \text{—} (5)$$

Now, eq. (4) and eq. (5) is coordinates of ellipse.

ASSIGNMENT - 2

UNIT - 2

"PARTIAL DIFFERENTATION"

Que. 1 If $u = (x^2+y^2+z^2)^{-1/2}$, $x^2+y^2+z^2 \neq 0$ then prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$.

Sol. ⇒

$$\frac{\partial^2 u}{\partial x^2} = -x\left(-\frac{3}{2}(x^2+y^2+z^2)^{-\frac{5}{2}} \cdot 2x\right) + \left[(x^2+y^2+z^2)^{-3/2} \cdot (-1)\right]$$

$$\frac{\partial^2 u}{\partial x^2} = 3x^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2}$$

$$\frac{\partial^2 u}{\partial x^2} = (x^2+y^2+z^2)^{-3/2}\left(\frac{3x^2}{x^2+y^2+z^2} - 1\right)$$

Similarly,

$$\frac{\partial^2 u}{\partial y^2} = (x^2+y^2+z^2)^{-3/2}\left(\frac{3y^2}{x^2+y^2+z^2} - 1\right)$$

$$\frac{\partial^2 u}{\partial z^2} = (x^2+y^2+z^2)^{-3/2}\left(\frac{3z^2}{x^2+y^2+z^2} - 1\right)$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = (x^2+y^2+z^2)^{-\frac{3}{2}}\left(\frac{3x^2}{x^2+y^2+z^2} + \frac{3y^2}{x^2+y^2+z^2} + \frac{3z^2}{x^2+y^2+z^2} - 3\right)$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = (x^2+y^2+z^2)^{-\frac{3}{2}}\left[\frac{3\cancel{(x^2+y^2+z^2)}}{\cancel{(x^2+y^2+z^2)}} - 3\right]$$

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0}$$

Hence Proved.

Date :
P. No. :

Que. 2 If $u = e^{xyz}$, Prove that

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = (1 + 3xyz + x^2y^2z^2)e^{xyz}.$$

Take L.H.S,

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}\right)\right]$$

$$= \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left(e^{xyz} \cdot xy\right)\right]$$

$$= \frac{\partial}{\partial x}\left[(e^{xyz} \cdot xz)xy + (e^{xyz}) \cdot x\right]$$

$$= \frac{\partial}{\partial x}\left[e^{xyz} \cdot x^2zy + e^{xyz} \cdot x\right]$$

$$= \frac{\partial}{\partial x}\left[e^{xyz}(x^2zy + x)\right]$$

$$= e^{xyz}(2zyx + 1) + \left[e^{xyz}(yz)(x^2zy + x)\right]$$

$$= e^{xyz}\left[2xyz + 1 + yz(x^2zy + x)\right]$$

$$= e^{xyz}(2xyz + 1 + x^2y^2z^2 + xyz)$$

$$= e^{xyz}(1 + 3xyz + x^2y^2z^2)$$

Hence Proved

Que. 3 If $u = \log(x^3 + y^3 + z^3 - 3xyz)$

Prove that (i) $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}$

(ii) $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 \cdot u = \frac{-9}{(x+y+z)^2}$.

(i) $\frac{\partial u}{\partial x} = \frac{1}{x^3+y^3+z^3-3xyz}(3x^2 - 3yz)$

$\frac{\partial u}{\partial y} = \frac{1}{x^3+y^3+z^3-3xyz}(3y^2 - 3zx)$

$\frac{\partial u}{\partial z} = \frac{1}{x^3+y^3+z^3-3xyz}(3z^2 - 3xy)$

$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{1}{x^3+y^3+z^3-3xyz}\left[3x^2+3y^2+3z^2-3xy-3yz-3zx\right]$

$= \frac{3}{x^3+y^3+z^3-3xyz}\left[x^2+y^2+z^2-xy-yz-zx\right]$

[# FORMULA USED ⇒ $x^3+y^3+z^3-3xyz = (x+y+z)(x^2+y^2+z^2-xy-yz-zx)$]

$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{(x+y+z)} \cancel{\frac{(x^2+y^2+z^2-xy-yz-zx)}{(x^2+y^2+z^2-xy-yz-zx)}}$

$\boxed{\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{(x+y+z)}}$ Hence Proved..

(ii) $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u=$

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$$

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$$

$$\frac{-3}{(x+y+z)^2}-\frac{3}{(x+y+z)^2}-\frac{3}{(x+y+z)^2}$$

$$=\frac{-9}{(x+y+z)^2}$$ Hence Proved..

Que. 4 If $x^x y^y z^z = C$ then prove that

$$\frac{\partial^2 z}{\partial x \partial y} = -(x \log ex)^{-1}.$$

Sol.^{n} ⇒ Given $x^x y^y z^z = C$ — (1)

Taking logarithm on both side, we get

$$x \log x + y \log y + z \log z = \log c \quad — (2)$$

eq. (2) partially diffrentiating w.r.t x by nothing that z is the func.^{n} of x & y.

$$\left[\log x + x\cdot\frac{1}{x}\right] + 0 + \left[\log z\cdot\frac{\partial z}{\partial x} + z\cdot\frac{1}{z}\frac{\partial z}{\partial x}\right] = 0$$

$$[\log x + 1] + [\log z + 1]\frac{\partial z}{\partial x} = 0$$

$$\boxed{\frac{\partial z}{\partial x} = \frac{-[1+\log x]}{[1+\log z]}} \quad (3)$$

ilarly,

$$\boxed{\frac{\partial z}{\partial y} = \frac{-[1+\log y]}{[1+\log z]}} \quad (4)$$

$$\frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = -\frac{\partial}{\partial x}\left[\frac{1+\log y}{1+\log z}\right]$$

$$= -(1+\log y)\frac{\partial}{\partial x}(1+\log z)^{-1}$$

$$= -(1+\log y)(-1)(1+\log z)^{-2}\left(\frac{1}{z}\right)\frac{dz}{dx}$$

$$= \frac{1+\log y}{z(1+\log z)^2}\left\{\frac{(1+\log x)}{(1+\log z)}\right\} \text{ from eq. } (4)$$

$$\boxed{\text{At } x=y=z}$$

$$\frac{\partial^2 z}{\partial x\,\partial y} = \frac{-(1+\log x)^2}{x(1+\log x)^3} = \frac{-1}{x(1+\log x)}$$

$$= \frac{-1}{\quad} = -(x\log ex)^{-1}$$

Que. 5 If $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$, then Prove that

(i) $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \sin 4u - \sin 2u.$

$$\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u} \quad \text{--- (1)}$$

Differentiate w.r.t. x

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial y \partial x} = \cos 2u \cdot 2\frac{\partial u}{\partial x}$$

$$\boxed{x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial y \partial x} = (2\cos 2u - 1)\frac{\partial u}{\partial x}} \quad \text{--- (2)}$$

Diff. eq. (1) w.r.t y

$$\boxed{\frac{x\partial^2 u}{\partial x \partial y} + y\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = 2\cos 2u \cdot \frac{\partial u}{\partial y}} \quad \text{--- (3)}$$

$$\frac{x^2\partial^2 u}{\partial x \partial y} + y\frac{\partial^2 u}{\partial y^2} = 2\cos 2u \cdot \frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}$$

$$\frac{x^2\partial^2 u}{\partial x \partial y} + y\frac{\partial^2 u}{\partial y^2} = (2\cos 2u - 1)\frac{\partial u}{\partial y}$$

Date :
P. No. :

iat

1.

Multiply eq. (2) by x & eq. (3) by y & Add

$$x^2\frac{\partial^2 u}{\partial x^2} + xy\frac{\partial^2 u}{\partial y \partial x} + xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = (2\cos 2u - 1)\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

$$x^2\frac{\partial^2 u}{\partial x^2} + \frac{2xy\,\partial^2 u}{\partial x\,\partial y} + \frac{y^2\partial^2 u}{\partial y^2} = (2\cos 2u - 1)\sin 2u$$

$$= 2\cos 2u \cdot \sin 2u - \sin 2u$$

$$= \sin 4u - \sin 2u$$

$$= 2\cos 3u \sin u$$

Hence Proved

Que. 6 If $u = \dfrac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}}$, Verify the Euler's theoram?

Sol.^{n} ⇒ Here, $u = \dfrac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}}$

Thus, u is a homogenous. $n = 1/20$
func.^{n} of x & y of degree $1/20$.

Hence to verify Euler's theoram for u we have to prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{1}{20}u$

Date :
P. No. :

diff. w.r.t (x)

$$\frac{\partial u}{\partial x} = \frac{(x^{1/5}+y^{1/5})\left(\frac{1}{4}x^{-3/4}\right) - (x^{1/4}+y^{1/4})\left(\frac{1}{5}x^{-4/5}\right)}{(x^{1/5}+y^{1/5})^2}$$

$$\frac{\partial u}{\partial x} = (x^{1/5}+y^{1/5})\left(\frac{1}{4}x^{-1}x^{-1/4}\right) - (x^{1/4}+y^{1/4})\left(\frac{1}{5}x^{-1}x^{-1/5}\right)$$

$$\frac{x\partial u}{\partial x} = \frac{(x^{1/5}+y^{1/5})\left(\frac{1}{4}x^{1/4}\right) - (x^{1/4}+y^{1/4})\left(\frac{1}{5}x^{1/5}\right)}{(x^{1/5}+y^{1/5})^2}$$

Again eq.-(2) partially diff. w.r.t (y)

$$\frac{\partial u}{\partial y} = \frac{(x^{1/5}+y^{1/5})\left(\frac{1}{4}y^{-3/4}\right) - (x^{1/4}+y^{1/4})\left(\frac{1}{5}y^{-\frac{4}{5}}\right)}{(x^{1/5}+y^{1/5})^2}$$

$$\frac{\partial u}{\partial y} = (x^{1/5}+y^{1/5})\left(\frac{1}{4}\cdot y^{-1}y^{-\frac{1}{4}}\right) - (x^{1/4}+y^{1/4})\left(\frac{1}{5}y^{-1}y^{-\frac{1}{5}}\right)$$

$$y\frac{du}{dy} = \frac{(x^{1/5}+y^{1/5})\left(\frac{1}{4}y^{1/4}\right) - (x^{1/4}+y^{1/4})\left(\frac{1}{5}y^{1/5}\right)}{(x^{1/5}+y^{1/5})^2} \quad \text{---} (4)$$

Adding eq. (3) & (4)

$$\therefore \frac{xdu}{dy} + \frac{ydu}{dy} = \frac{(x^{1/5}+y^{1/5})\left(\frac{1}{4}x^{1/4}\right) + (x^{1/5}+y^{1/5})\left(\frac{1}{4}x^{1/4}\right) \left[-(x^{1/4}+y^{1/4})\left(\frac{1}{5}y^{1/5}\right)\right]}{(x^{1/5}+y^{1/5})^2}$$

 Piyush

$$x\frac{du}{dx} + y\frac{du}{dy} = \frac{(x^{1/5}+y^{1/5})\left(\frac{1}{4}x^{1/4}+\frac{1}{4}y^{1/4}\right) - (x^{1/4}+y^{1/4})(x^{1/5}+y^{1/5})}{(x^{1/5}+y^{1/5})^2}$$

$$x\frac{du}{dx} + y\frac{du}{dy} = \frac{\cancel{(x^{1/5}+y^{1/5})}(x^{1/4}+y^{1/4})\left(\frac{1}{4}-\frac{1}{5}\right)}{(x^{1/5}+y^{1/5})^{\cancel{2}}}$$

$$\boxed{x\frac{du}{dx} + y\frac{du}{dy} = \frac{[x^{1/4}+y^{1/4}]}{[x^{1/5}+y^{1/5}]}\left[\frac{1}{20}\right]}$$ Ans.

Que. 7 If $u = \sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u$.

Sol.^{n} ⇒ $\sin u = \frac{x^2+y^2}{x+y} = \frac{x^{\cancel{2}}}{\cancel{x}}\left[\frac{(1+(\frac{y}{x})^2)}{1+\frac{y}{x}}\right]$ $\boxed{n=1}$

$f = \sin u$

By Euler's theorem

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$$

$$x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = \sin u$$

$$\cos u\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right] = \sin u$$

$$\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{\sin u}{\cos u} = \tan u}$$ Hence Proved

L.H.S = R.H.S

Date :
P. No. :

Que. 8 Evaluate cube root 127 approximately.

Sol.^{n} ⇒

Let, $y = f(x) = \sqrt[3]{x} = x^{1/3}$

$$f'(x) = \frac{1}{3} x^{-\frac{2}{3}}$$

$$\boxed{\delta y = f(x+\delta x) - f(x)}$$

$$\delta y = f'(x) \cdot \delta x$$

$$f(x+\delta x) - f(x) = f'(x) \cdot \delta x$$

Now, $\left[\because x = 125 \ \& \ \delta x = 2 \right]$

$$f(125+2) - \sqrt[3]{125} = \frac{1}{3}(125)^{-2/3}(2)$$

$$f(127) = 5 + \frac{2}{3} \times \frac{1}{25}$$

$$f(127) = 5 + \frac{2}{75}$$

$$\boxed{f(127) = 5.026666667}$$ Ans.